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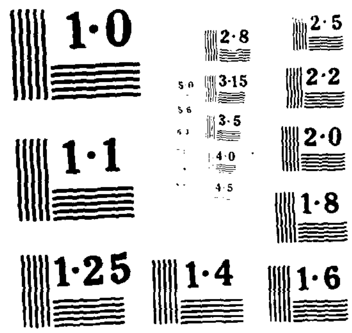
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OPERATIONS RESEARCH AND SYSTEMS ANALYSIS

Queueing Analysis of Fault-Tolerant Computer Systems*

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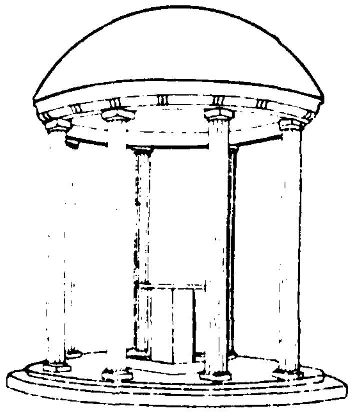
K.S. Trivedi‡

Technical Report No. UNC/ORSA/TR-85/10

UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL



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In this paper we analyze a fault-tolerant computer system. The failure/repair behavior of the system is modelled by an irreducible continuous-time

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Markov chain. Jobs arrive in a Poisson fashion to the system and are serviced according to an FCFS discipline. A failure may cause the loss of the work already done on the job in service, if any; in this case the iterrupted job is repeated as soon as the system is ready to deliver service. In addition to the delays due to failures and repairs, jobs suffer delays due to queueing. We present a general queueing analysis of fault-tolerant systems and study the steady-state behavior of the number of jobs in the system. As a numerical example, we consider a system with two processors subject to failures and repairs.

Abstract

In this paper we analyze a fault-tolerant computer system. The failure/repair behavior of the system is modelled by an irreducible continuous-time Markov chain. Jobs arrive in a Poisson fashion to the system and are serviced according to an *FCFS* discipline. A failure may cause the loss of the work already done on the job in service, if any; in this case the interrupted job is repeated as soon as the system is ready to deliver service. In addition to the delays due to failures and repairs, jobs suffer delays due to queueing. We present a general queueing analysis of fault-tolerant systems and study the steady-state behavior of the number of jobs in the system. As a numerical example, we consider a system with two processors subject to failures and repairs.

A-1

1. Introduction

Queueing models provide a useful tool for predicting the performance of many service systems including computer systems, telecommunication systems, computer/communication networks and flexible manufacturing systems. Traditional queueing models predict system performance under the assumption that all service facilities provide failure-free service [7]. It must, however, be acknowledged that service facilities do experience failures and that they get repaired. Failure/repair behavior of such systems is commonly modelled separately using techniques classified under reliability/availability modeling [3]. In recent years, it has been increasingly recognized that this separation of performance and reliability/availability models is no longer adequate [13].

Two distinct approaches towards combined modeling of performance and reliability/availability have been used. In the first approach, queueing models with server breakdowns and repairs are analyzed by means of generating functions [14], supplementary variables [2], imbedded Markov process and renewal theory [6], or probabilistic [17] techniques. These efforts generally carry out an exact steady-state queueing analysis of the system in the presence of breakdowns and repairs. A transient analysis of an M/G/1 queue with server breakdown, subject to hard-deadline constraint on response time, has been considered recently [1]. The second approach is approximate, in which it is assumed that the time to reach the steady-state is much smaller than the times to failures/repairs. Therefore, it is reasonable to associate a performance measure (reward) with each state of the underlying Markov (or semi-Markov) model describing the failure/repair behavior of the system. Each of these performance measures is obtained from the steady-state queueing analysis of the system in the corresponding state. The resulting reward model is then analyzed for the expected values or the distributions of interesting cumulative measures of system performance [8,9,10,13].

Since the job oriented view of performance/reliability in fault-tolerant system is particularly important, models have been developed to derive the distribution of job completion time in a failure-prone environment. In these models, we need to consider a possible loss of work due to the occurrence of a failure, i.e., the interrupted job may be resumed or restarted upon service resumption. We have recently considered models that take into account different types of interruptions in the analysis of the job completion time [8,9,10]. These earlier results are the bases for the queueing analysis presented in this paper. Note that the job completion time analysis includes the delays due to failures and repairs, but it does not account for queueing delays. The purpose of this paper is to extend our earlier analysis so as to account for the queueing delays. In effect, we consider an exact queueing analysis of the system in order to obtain the steady-state distribution and the mean of the number of jobs in the system.

We consider a queueing model of a computer system where the jobs arrive in a Poisson fashion with rate λ. The service requirements of the incoming jobs form a sequence of independent and identically distributed random variables with common cdf $G(.)$. The computer system exists in one of n possible states. The state of the computer system changes with time according to an independent continuous-time Markov chain. It is assumed that this chain is irreducible. When the computer system is in state i it delivers service at rate $r_i \geq 0$. A state i may be classified as preemptive-resume (*prs*) or preemptive-repeat-identical (*pri*) as follows: A state is said to be *prs* (*pri*) if, upon entering that state, the work done so far on the current job is preserved (lost), and the service is resumed (restarted) in the new state. Thus the actual time required to complete a job depends in a complex way upon the service requirement of the job and the evolution of the state of the computer system. It is assumed that there is an infinite waiting room for the jobs and that the service discipline is first come first served (*FCFS*). Note that even though the service requirements of jobs are independent and identically distributed, the actual times required to complete these jobs are neither independent nor identically distributed, and hence the model cannot be reduced to a standard M/G/1 queue [17].

When all the states of the model describing the computer system are *prs*, i.e., no work is ever lost, the problem described here can be analyzed as queues in random environments (e.g., see Purdue [18]). In the present paper we carry the analysis with the possibility of work loss, i.e., when some of the states of the underlying Markov model are *prs* and the remaining *pri*. As loss of work due to failures and interruptions is quite a common phenomenon in fault-tolerant computer systems, the model proposed here is of obvious interest. Many of the breakdown-repair queueing models studied in the literature are special cases of the model studied here (e.g., Mitrani [14], Nicola [17], Baccelli and Trivedi [2] and others).

In the next section we first study the situation with no queueing and state some results, concerning the analysis of job completion time, which are direct extension of those given in [9]. Using these results we set up the queueing model in Section 3 and show that it has the block M/G/1 structure. Queueing models with such a structure have been studied by Neuts [15], Neuts and Lucantoni [16], Ramaswami [19] and others. We demonstrate the usefulness of our approach by performing the numerical analysis for a particular example. This is done in Section 4. Finally, conclusions and some extensions are discussed in Section 5.

2. *The Completion Time Analysis of a Single Job*

Consider a single job with service requirement B that starts getting served at time 0. B is a random variable with a distribution function $G(x) = P(B \leq x)$ and LST $G^{*}(.)$. Let $Z(t)$ be the state of the computer system at time t. It is assumed that $\{Z(t), t \geq 0\}$ is an irreducible continuous-time Markov chain on $\{1,2,...,n\}$ with $n \times n$ generator matrix $Q = [q_{ij}]$, where q_{ij}, $(i \neq j)$ is the transition rate from state i to state j, and $q_{ii} = -\sum_{j \neq i} q_{ij}$. Thus, each row of Q sums to zero. Let T be the time when this job completes its service. Define, for $1 \leq i, j \leq n$,

$$F_{ij}(t,x) = P(T \leq t, Z(T)=j \mid Z(0)=i, B=x),$$

$$F_{ij}^{*}(s,x) = E(e^{-sT}; Z(T)=j \mid Z(0)=i, B=x) = \int_0^{\infty} e^{-st}\, dF_{ij}(t,x) \ ,$$

$$F_{ij}^{**}(s,w) = \int_0^{\infty} e^{-wx} F_{ij}^{*}(s,x)dx.$$

Theorems 2.1 and 2.2 below are minor extensions of theorems 2 and 3 in [8]. They treat the cases where all the states are *prs* or all the states are *pri*, respectively. The proofs are similar to those in [8], and hence, are omitted.

Theorem 2.1. Let all the states be of the *prs* type. The double transforms $F_{ij}^{**}(s,w)$, $1 \leq i,j \leq n$, are given by the unique solution to

$$F_{ij}^{\sim *}(s,w) = \frac{r_i \delta_{ij}}{s+q_i+r_i w} + \sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{q_{ik}}{s+q_i+r_i w} F_{kj}^{\sim *}(s,w), \quad 1 \leq i, j \leq n. \tag{2.1}$$

where $q_i = -q_{ii}$, $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

Theorem 2.2. Let all the states be of the *pri* type. The transforms $F_{ij}^{\sim}(s,x)$, $1 \leq i, j \leq n$, are given by the unique solution to

$$F_{ij}^{\sim}(s,x) = e^{-(s+q_i)x/r_i}\delta_{ij} + \sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{q_{ik}}{s+q_i}(1-e^{-(s+q_i)x/r_i})F_{kj}^{\sim}(s,x), \quad 1 \leq i, j \leq n. \tag{2.2}$$

Next we consider the mixed case. Let $S \subset \{1,2,\ldots,n\}$ be the set of *pri* states, and $\overline{S} = \{1,2,\ldots,n\} - S$ be the set of *prs* states. Suppose $Z(0) \in \overline{S}$ and let

$$U = \min\{t \geq 0 : Z(t) \in S\}.$$

Define

$$V = \min\{T, U\}.$$

For $i \in \overline{S}$, define

$$M_{ij}(t,x) = P(V \leq t, Z(V) = j \mid Z(0) = i, B = x),$$

with the corresponding LST $M_{ij}^{\sim}(s,x)$, and the double transform

$$M_{ij}^{\sim *}(s,w) = \int_0^\infty e^{-wx} M_{ij}^{\sim}(s,x)\,dx.$$

Note that

$$M_{ij}(t,x) = P(T \leq t < U, Z(T) = j \mid Z(0) = i, B = x), \text{ if } j \in \overline{S}$$

and

$$M_{ij}(t,x) = P(U \leq t < T, Z(U) = j \mid Z(0) = i, B = x), \text{ if } j \in S.$$

The following theorem is an extension of propositions 5.1 and 5.2 in [9]; the proof follows along the same lines, and hence is omitted.

Theorem 2.3. Let S be the set of *pri* states and $\overline{S}$ be the set of *prs* states. Then

(i) The double transforms $M_{ij}^{\sim *}(s,w)$, $i,j \in \overline{S}$, satisfy the following equations

$$M_{ij}^{\sim *}(s,w) = \frac{r_i \delta_{ij}}{s+q_i+r_i w} + \sum_{\substack{k \in \overline{S} \\ k \neq i}} \frac{q_{ik}}{s+q_i+r_i w} M_{kj}^{\sim *}(s,w), \quad i,j \in \overline{S} \tag{2.3}$$

(ii) The double transforms $M_{ij}^{\sim *}(s,w)$, $i \in \overline{S}$, $j \in S$, satisfy the following equations

$$M_{ij}^{\sim *}(s,w) = \frac{q_{ij}}{w(s+q_i+r_i w)} + \sum_{\substack{k \in \overline{S} \\ k \neq i}} \frac{q_{ik}}{s+q_i+r_i w} M_{kj}^{\sim *}(s,w), \quad i \in \overline{S}, j \in S \tag{2.4}$$

Equations (2.3) and (2.4) have unique solutions.

The following theorem gives a method for determining $F_{ij}^{\sim}(s,x)$, $1 \leq i, j \leq n$, which is the main result of this section. The proof of this theorem, being similar to those of theorems 5.1 and 5.2 of [9], is omitted.

Theorem 2.4. Let the states in S be *pri* and those in $\overline{S}$ be *prs*. Then

(i) The *LST*s $F_{ij}^{\sim}(s,x)$, $i \in S$, satisfy the following equations

$$F_{ij}^{\sim}(s,x) = g_{ij}(s,x) + \sum_{l \in S} h_{il}(s,x) F_{lj}^{\sim}(s,x), \; i \in S, \; 1 \leq j \leq n. \tag{2.5}$$

where

$$g_{ij}(s,x) = \begin{cases} \delta_{ij} e^{-(s+q_i)x/r_i} + \sum_{k \in \overline{S}} \sum_{l \in \overline{S}} \delta_{lj} \frac{q_{ik}}{r_i} \int_0^x e^{-(s+q_i)h/r_i} M_{kl}^{\sim}(s,x-h)\,dh, & \text{if } r_i > 0 \\ \sum_{k \in \overline{S}} \sum_{l \in \overline{S}} \delta_{lj} \frac{q_{ik}}{(s+q_i)} M_{kl}^{\sim}(s,x), & \text{if } r_i = 0, \quad i \in S, 1 \leq j \leq n \end{cases}$$

and

$$h_{il}(s,x) = \begin{cases} \frac{q_{il}}{(s+q_i)}(1-e^{-(s+q_i)x/r_i}) + \sum_{k \in \overline{S}} \frac{q_{ik}}{r_i} \int_0^x e^{-(s+q_i)h/r_i} M_{kl}^{\sim}(s,x-h)\,dh, & \text{if } r_i > 0 \\ \frac{q_{il}}{(s+q_i)} + \sum_{k \in \overline{S}} \frac{q_{ik}}{(s+q_i)} M_{kl}^{\sim}(s,x), & \text{if } r_i = 0, \quad i,l \in S. \end{cases}$$

(ii) The LST s $F_{ij}^{*}(s,x)$, $i \in \overline{S}$, are given by

$$F_{ij}^{*}(s,x) = \sum_{l \in \overline{S}} \delta_{lj} M_{il}^{*}(s,x) + \sum_{l \in S} M_{il}^{*}(s,x) F_{lj}^{*}(s,x),\ i \in \overline{S},\ 1 \le j \le n. \tag{2.6}$$

Thus, for the mixed case, the job completion time is completely described by the LST s $F_{ij}^{*}(s,x)$, given by theorem 2.4. These expressions are essential for the queueing analysis in the next section.

3 The Queueing Model

In this section we perform the steady state analysis of the queueing model described in the introduction. Let $X(t)$ be the number of jobs in the system (including any in service) at time t. Let τ_ν be the time when the ν-th job is completed. Assume that $\tau_0 = 0$ and a new job starts service at time 0.

Let $X_\nu = X(\tau_\nu+)$ and $Z_\nu = Z(\tau_\nu+)$ be the number of jobs and the state of the system, respectively, immediately after the ν-th job completion. Due to the Poisson arrivals and the Markov nature of $\{Z(t), t \ge 0\}$, it is clear that $\{(X_\nu, Z_\nu), \nu \ge 0\}$ is a discrete time Markov chain with state space $\{0,1,\ldots\} \times \{1,2,\ldots,m\}$ where $m = |\{i : r_i > 0\}|$, $m \le n$. (Note that a job may complete in state i only if $r_i > 0$). In this section we study the limiting distribution of $\{(X_\nu, Z_\nu), \nu \ge 0\}$. The relevance of this limiting distribution follows from the fact that jobs arrive singly to the system and depart singly from the system, and that the arrival process is Poisson. Therefore,

$$\lim_{t \to \infty} P(X(t)=j) = \lim_{\nu \to \infty} P(X_\nu = j)$$

when the limits exist. This is a well known theorem (see Cooper [5]).

Next we determine the one step transition probability matrix of $\{(X_\nu, Z_\nu), \nu \ge 0\}$.

3.1. The Transition Probability Matrix

We first note that a job may start while the system is in any state, but it may complete only if the system is in a state with a positive service rate. From now on, we assume that $r_i > 0$ for $1 \le i \le m$ and $r_i = 0$ for $m < i \le n$. Assume that the unconditional LST

$$F_{ij}^{*}(s) = \int_0^\infty F_{ij}^{*}(s,x)\, dG(x),\ 1 \le i \le n,\ 1 \le j \le m,$$

has been computed using the methods in Section 2. $F_{ij}(t)$ denotes the inverse of $\tilde{F}_{ij}(s)$, i.e.,

$$F_{ij}(t) = P(T \le t, Z(T) = j \mid Z(0) = i),\ 1 \le i \le n,\ 1 \le j \le m.$$

Now let

$$a_{ij}(k) = P(Z(T) = j,\ number\ \ of\ \ arrivals\ \ during\,(0,T] = k \mid Z(0) = i)$$

$$= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!}\, dF_{ij}(t),\quad k = 0,1,2,\ldots,\ 1 \le i \le n,\ 1 \le j \le m.$$

Define the $n \times m$ matrix $A^*(k) = [a_{ij}(k)], k \ge 0$. Now, let Y be an exponentially distributed random variable with parameter λ, which is independent of $\{Z(t), t \ge 0\}$. The following quantity will be needed in our analysis

$$d_{ij} = P(Z(Y) = j \mid Z(0)=i)$$

$$= \lambda \int_0^\infty e^{-\lambda t} P(Z(t)=j \mid Z(0)=i)dt,\ 1 \le i,\ j \le n.$$

d_{ij} is the probability that the system is in state j at the time of the next arrival, given that the system was empty in state i. Recognizing the integral as the Laplace transform we get the following formula for the $n \times n$ matrix $D\,(= [d_{ij}])$

$$D = \lambda[\lambda I - Q]^{-1}$$

where Q is the generator matrix of $\{Z(t), t \ge 0\}$ as defined in section 2. Using above notation we give the following theorem.

Theorem 3.1.

$$P(X_{\nu+1}=k',Z_{\nu+1}=j \mid X_\nu=k, Z_\nu=i)$$

$$= \begin{cases} a_{ij}(k'-k+1), & \text{if } k' \ge k-1 \ge 0 \\ \sum_{l=1}^{n} d_{il}\, a_{lj}(k'), & \text{if } k' \ge k = 0 \\ 0, & otherwise, \qquad 1 \le i,j \le m \end{cases}$$

Proof.

(i) Let $k' \geq k-1 \geq 0$. Then $X_{\nu+1} = X_\nu - 1 +$ number of arrivals during $(\tau_\nu, \tau_{\nu+1})$. The service of the $(\nu+1)$-th customer starts at time τ_ν with the system in state i and ends with the system in state j. Hence the required conditional probability is $a_{ij}(k' - k + 1)$.

(ii) Let $k' \geq k = 0$. Thus the system is empty when the ν–th job completes and the system is in state i. There follows an idle period Y of exponential duration with parameter λ, during which time the state of the system changes to l with probability d_{il}. The service of the $(\nu+1)$-th job starts in state l and $X_{\nu+1}=$ number of arrivals during this service time. Hence the required conditional probability is given by $\sum_{l=1}^{n} d_{il}\, a_{lj}(k')$, and is denoted by $b_{ij}(k')$, $1 \leq i,j \leq m$.

Let $F(t) = [F_{ij}(t)]$ $(i=1,2,...,n\,;\; j=1,2,...,m)$ be a $n \times m$ matrix and $F_{(red)}(t)$ be a $m \times m$ submatrix of it obtained by taking its first m rows. For $k \geq 0$, define the $n \times m$ matrices $A^*(k)$ and $m \times m$ matrices $A(k)$ as follows:

$$A^*(k) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} dF(t)$$

$$A(k) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k} dF_{(red)}(t).$$

Also define $m \times m$ matrices B(k), $k \geq 0$, as follows

$$B(k) = D_{(red)} A^*(k)$$

where $D_{(red)}$ is the $m \times n$ submatrix of $D = \lambda(\lambda I - Q)^{-1}$ obtained by deleting its last $n-m$ rows.

Define the macro state vector $\underline{i} = \{(i,1),(i,2),...,(i,m)\}$, $i=0,1,2,....$ The macro state $\underline{i}$ means that there are i jobs in the system, upon an arbitrary job completion. Using the state space $\{\underline{i} : i \geq 0\}$, we can write the one-step transition probability matrix of $\{(X_\nu, Z_\nu), \nu \geq 0\}$, as

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \end{array} \\ \left[\begin{array}{ccccc} B(0) & B(1) & B(2) & B(3) & \cdots \\ A(0) & A(1) & A(2) & A(3) & \cdots \\ & A(0) & A(1) & A(2) & \cdots \\ & & A(0) & A(1) & \cdots \\ & & & A(0) & \cdots \end{array}\right] \end{array}$$

Notice that Z_ν can be in state i if and only if $r_i > 0$. Therefore, $A(k)$ and $B(k), k \geq 0$, are all $m \times m$ matrices, and all elements of $A(k)$ and $B(k), k \geq 0$, are strictly positive.

From the above matrix representation, it is obvious that our model has a block M/G/1 structure. As mentioned before, there are a large number of models that fall into this structure and a general algorithm for the solution of this problem has been studied in detail by Lucantoni and Neuts [11], Lucantoni and Ramaswami [12] and Neuts [15]. Here, we use a modified version of the method of Lucantoni and Neuts [11].

Now let $A = \sum_{k=0}^{\infty} A(k) = \tilde{F}_{(red)}(0)$. Note that $A (=[a_{ij}])$ is an irreducible stochastic matrix. Let $\underline{\pi}$ be its invariant solution, i.e.,

$$\underline{\pi}^T = \underline{\pi}^T A, \quad \underline{\pi}^T \underline{e} = 1.$$

where $\underline{e}$ is an m-dimensional column vector of 1s. We define $\underline{\beta} = \sum_{k=1}^{\infty} kA(k)\underline{e} = [-\lambda \frac{d}{ds}\tilde{F}_{(red)}(s) \mid_{s=0}]\underline{e}$. Note that $\underline{\pi}^T \underline{\beta}$ is the expected number of arrivals per departure at saturation (i.e., assuming that the system is never empty). Hence the condition of stability for the queueing system is given by (Neuts [15]):

$$\rho = \underline{\pi}^T \underline{\beta} < 1,$$

which can be rewritten as

$$\lambda < \lambda^*$$

where λ^* is the threshold value of the job arrival rate below which the queueing system will remain stable. We assume that the above condition is satisfied so that the Markov chain $\{(X_\nu, Z_\nu), \nu \geq 0\}$ is positive recurrent.

Now let

$$y(i,j) = \lim_{\nu \to \infty} P(X_\nu = i, Z_\nu = j), \quad i \geq 0, 1 \leq j \leq m.$$

The infinite probability vector $\underline{y}^T$ is written as $(\underline{y}_0^T, \underline{y}_1^T, \underline{y}_2^T, \ldots)$, where $\underline{y}_i^T = [y(i,1), y(i,2), \ldots, y(i,m)]$. Define $\phi_j(w) = \sum_{i=0}^{\infty} w^i y(i,j)$ $(j=1,2,\ldots,m)$ and let

$\underline{\phi}^T(w) = (\phi_1(w), \phi_2(w), \ldots, \phi_m(w))$. Then it can be easily shown that

$$\underline{\phi}^T(w) = \underline{y}_0^T\,[w\hat{B}(w) - \hat{A}(w)][wI - \hat{A}(w)]^{-1} \tag{3.1}$$

where

$$\hat{B}(w) = \sum_{k=0}^{\infty} w^k B(k) = D_{(red)} F^{\hat{}}\ (\lambda(1-w)),$$

$$\hat{A}(w) = \sum_{k=0}^{\infty} w^k A(k) = F^{\hat{}}_{(red)}(\lambda(1-w)),$$

The standard procedure at this point is to determine $\underline{y}_0^T$ by complex function theory arguments based upon the holomorphic nature of $\underline{\phi}^T(w)$, but this procedure is numerically unstable. Lucantoni and Neuts [11] have developed a more stable procedure to obtain $\underline{y}_0^T$. We use a modified version of this procedure which is described here for completeness.

Let V be the length of a busy period initiated by a single customer at time 0. Define

$$H_{ij} = P\{Z(V)=j \mid Z(0)=i\}, \quad i,j = 1,2,\ldots,m,$$

to be the probability that the system state changes from i to j during a busy period. It is known that the matrix $H = [H_{ij}]$ is the smallest solution to the following nonlinear equation:

$$H = \sum_{k=0}^{\infty} A(k) H^k$$

$$= \int_0^{\infty} dF_{(red)}(t)\, exp\,(\lambda(H-I)t). \tag{3.2}$$

Equation (3.2) can be solved by a straightforward iterative method:

$$H_0 = I$$

$$H_{n+1} = \int_0^{\infty} dF_{(red)}(t) \exp(\lambda(H_n - I)t) \tag{3.3}$$

In the limit as n approaches infinity, H_n of equation (3.3) approaches H, the solution to (3.2). Notice that when the matrix exponential in the above equation is computed by an eigenvalue technique, the equation (3.3) gives H_{n+1} in terms of $F_{ij}^{\hat{}}(\cdot)$ evaluated at the eigenvalues of H_n. This method is used in the example of next section, and it obviates the need to compute $A(k)$ for all k. It should be noted that H is a stochastic matrix.

It is known that $\underline{y}_0^T$ is a solution to

$$\underline{y}_0^T = \underline{y}_0^T\left(\sum_{k=0}^{\infty} B(k)H^k\right) = \underline{y}_0^T \; D_{(red)} \int_0^{\infty} dF(t)\exp(\lambda(H-I)(t)). \tag{3.4}$$

The above equation determines $\underline{y}_0^T$ upto a multiplicative constant, since $\sum_{k=0}^{\infty} B(k)H^k$ is a stochastic matrix, and has rank $m-1$. Again, the matrix $\sum_{k=0}^{\infty} B(k)H^k$ can be computed in terms of $F_{ij}(\cdot)$ evaluated at the eigenvalues of H.

At this point, Lucantoni and Neuts provide a rather formidable procedure to compute this multiplicative constant. Here we provide an alternative method, which is based upon the following equation:

$$\lim_{w \to 1} \underline{\phi}^T(w)\underline{e} = 1. \tag{3.5}$$

Unfortunately, $wI-\hat{A}(w)$ is singular in the limit as $w \to 1$, and hence we need to use L'Hospital's rule to compute the limit in (3.5). To do this, write,

$$[wI-\hat{A}(w)]^{-1} = R(w)/u(w)$$

where $R(w)$ is the adjoint of $wI-\hat{A}(w)$ and $u(w)$ is the determinant of $wI-\hat{A}(w)$. Then we get

$$u'(1) = \underline{y}_0^T\{(\hat{B}(1)+\hat{B}'(1)-\hat{A}'(1))R(1)+(\hat{B}(1)-\hat{A}(1))R'(1)\}\underline{e} \tag{3.6}$$

where

$$\hat{A}(1) = F^{\sim}_{(red)}(0),\;\; \hat{B}(1) = D_{(red)}F^{\sim}(0),$$

$$\hat{A}'(1) = -\lambda\frac{d}{ds}F^{\sim}_{(red)}(s)\,|_{s=0},\;\; \hat{B}'(1) = D_{(red)}[-\lambda\frac{d}{ds}F^{\sim}(s)\,|_{s=0}].$$

Equation (3.6) above provides the required independent equation to determine the multiplicative constant. Once $\underline{y}_0^T$ is known, $\underline{\phi}^T(w)$ is completely determined and one can compute moments by taking derivatives.

One seeming difficulty of this procedure is the apparent necessity of having to compute $R(w)$ and $u(w)$ algebraically. As we only need $u(1), u'(1), R(1), R'(1)$, we can use the following theorems which eliminate the necessity of computing $R(w)$ and $u(w)$ algebraically.

Theorem 3.1. Let $G(w) = [G_{ij}(w)]$ be a $m \times m$ matrix of differentiable functions. For $k = 1,2,\ldots,m$, define $m \times m$ matrices $G^{(k)}(w)$ as follows:

$$[G^{(k)}(w)]_{ij} = \begin{cases} G_{ij}(w) & \text{if } i \neq k \\ \frac{d}{dw} G_{ij}(w) & \text{if } i = k \end{cases}$$

Then

$$\frac{d}{dw}(detG(w)) = \sum_{k=1}^{m} \det G^{(k)}(w).$$

Theorem 3.2. Let $G(w)$ and $G^{(k)}(w)$ be as defined in Theorem 3.1. For $k=1,2,\ldots,m$, define the $m \times m$ matrices $G_{(k)}(w)$ as follows:

$$[G_{(k)}(w)]_{ij} = \begin{cases} G_{ij}(w) & \text{if } i \neq k \\ 0 & \text{if } i = k \end{cases}$$

Then

$$\frac{d}{dw}[Adj(G(w))] = \sum_{k=1}^{m} [AdjG^{(k)}(w) - AdjG_{(k)}(w)].$$

Proofs of both these theorems are straightforward. These theorems provide $0(m^4)$ method of computing $u'(1)$ and $R'(1)$.

An interesting feature of this queueing model is that the expected service time of an arbitrary job in steady state depends upon the load offered to the system, viz. λ. We can easily derive expressions for this quantity when λ approaches 0 (a lightly loaded system) or as λ approaches λ^* (a heavily loaded system). Let S_ν be the service time of the ν^{th} customer. When the arrival rate $\lambda \to 0$, every incoming job finds the system empty. Thus, in steady state, an incoming job finds the structure state process in state i with probability $\theta_i = \lim_{t \to \infty} P(Z(t)=i)$. Hence

$$\lim_{\lambda \to 0} \lim_{\nu \to \infty} E(S_\nu) = \sum_{i=1}^{n} \theta_i E(T(x) \mid Z(0)=i)$$

where $\underline{\theta}^T = (\theta_1, \theta_2, \ldots, \theta_n)$ is a solution to

$$\underline{\theta}^T Q = 0, \ \underline{\theta}^T \underline{e} = 1,$$

and

$$E(T(x) \mid Z(0)=i) = -\frac{d}{ds}\sum_{j=1}^{n} F_{ij}^{*}(s,x)\,|_{s=0}.$$

When the arrival rate $\lambda \to \lambda^*$, the system is always busy. Hence, when a job comes up for service, the structure state process is in state i with probability, π_i, where $\underline{\pi}^T = (\pi_1, \ldots, \pi_m)$ is the stationary probability vector of the matrix A as defined before. Hence

$$\lim_{\lambda \to \lambda^*} \lim_{\nu \to \infty} E(S_\nu) = \sum_{i=1}^{m} \pi_i E(T(x) \mid Z(0)=i) = \frac{1}{\lambda^*}$$

In the next section we tabulate $\lim_{\nu \to \infty} E(S_\nu)$ as a function of λ for a two processor fault tolerant system. We also study the expected queue length in such a system as a function of λ.

4. An Example

In this section we consider an example to demonstrate the use of the techniques presented in section 3. We obtain the mean of the number of jobs in a fault-tolerant computer system in steady state. The system has two processor units subject to failures and repairs. The failure rate of a single processor is γ. The failure of one processor causes the preemption of the job being processed. The interrupted job is restarted and processed at a reduced service rate (service rate is assumed to be proportional to the number of operating processors). When both processors have failed, the interrupted job is restarted as soon as one of the processors is repaired and is processed at a reduced service rate. When the second processor is repaired the processing of the job is continued at increased (normal) service rate. The failed processors are repaired one at a time with a rate μ.

The behavior of the system can be described by a continuous-time Markov chain with the state-transition diagram shown in figure 1. Note that state 2 corresponds to the system with two operating processors, and is classified as a *prs* state. The service rate in state 2 is $r_2(=2)$. State 1 corresponds to the system with one operating processor, and is classified as a *pri* state. The service rate in state 1 is $r_1(=1)$. State 3 corresponds to the system with both processors failed, and is classified as a *pri* state. The service rate in state 3 is $r_3(=0)$. Jobs arrive into the system according to a Poisson process at a rate λ. Each job has a deterministic work requirement, say x units of work.

We now follow the procedure suggested in section 2 in order to compute $F_{ij}^{\cdot}(s,x)$, $i=1,2,3$; $j=1,2$. (Note that a job may complete only in a state with nonzero service rate). In our example there are two types of system states, the *prs* subset $\overline{S} = \{2\}$, and the *pri* subset $S = \{1,3\}$. From theorem 2.3, equations (2.3) and (2.4) yield

$$M_{22}^{\cdot *}(s,w) = \frac{2}{s+2\gamma+2w} \tag{4.1}$$

and

$$M_{21}^{\cdot *}(s,w) = \frac{2\gamma}{w(s+2\gamma+2w)}. \tag{4.2}$$

Inverting with respect to w, we get

$$M_{22}^{\cdot}(s,x) = e^{-(s+2\gamma)x/2} \tag{4.3}$$

and

$$M_{21}^{\cdot}(s,x) = (\frac{2\gamma}{s+2\gamma})(1-e^{-(s+2\gamma)x/2}). \tag{4.4}$$

The LSTs $F_{ij}^{\cdot}(s,x)$, $i=1,3; j=1,2$, can be determined from theorem 2.4, equation (2.5), as follows:

$$F_{31}^{\cdot}(s,x) = (\frac{\mu}{s+\mu})F_{11}^{\cdot}(s,x), \tag{4.5}$$

$$F_{32}^{\cdot}(s,x) = (\frac{\mu}{s+\mu})F_{12}^{\cdot}(s,x) \tag{4.6}$$

where

$$F_{11}^{\cdot}(s,x) = \frac{e^{-(s+\gamma+\mu)x}}{D(s,x)} \tag{4.7}$$

and

$$F_{12}^{\cdot}(s,x) = \frac{(\frac{2\mu}{s+2\mu})(e^{-(s+2\gamma)x/2}-e^{-(s+\gamma+\mu)x})}{D(s,x)} \tag{4.8}$$

with

$$D(s,x) = \left[1 - \frac{\gamma\mu(3s+2\gamma+2\mu)\,(1-e^{-(s+\gamma+\mu)x})}{(s+\mu)(s+2\gamma)(s+\gamma+\mu)} + \frac{4\gamma\mu\,(e^{-(s+2\gamma)x/2}-e^{-(s+\gamma+\mu)x})}{(s+2\gamma)(s+2\mu)}\right].$$

The LST s $F_{ij}^{\sim}(s,x)$, $i=2$, $j=1,2$, follow from theorem 2.4, equation (2.6), as follows

$$F_{21}^{\sim}(s,x) = M_{21}^{\sim}(s,x)F_{11}^{\sim}(s,x), \tag{4.9}$$

$$F_{22}^{\sim}(s,x) = M_{22}^{\sim}(s,x) + M_{21}^{\sim}(s,x)F_{12}^{\sim}(s,x) \tag{4.10}$$

with $M_{22}^{\sim}(s,x)$ and $M_{21}^{\sim}(s,x)$ as given by equations (4.3) and (4.4), respectively.

Now that we have evaluated the LST s $F_{ij}^{\sim}(s,x)$, $i=1,2,3$; $j=1,2$, we can carry out the queueing analysis as suggested in section 3. The matrix A is given by

$$A = \sum_{k=0}^{\infty} A(k)$$

$$= [F_{ij}^{\sim}(0,x)], \quad i,j=1,2$$

can now be evaluated using equations (4.7)-(4.10). It follows that

$$A = \begin{bmatrix} e^{-\mu x} & (1-e^{-\mu x}) \\ (e^{-\mu x}-e^{-(\gamma+\mu)x}) & (1-e^{-\mu x}+e^{-(\gamma+\mu)x}) \end{bmatrix}$$

As noted earlier, A is an irreducible stochastic matrix. Let $\underline{\pi}$ be the solution to $\underline{\pi}^T = \underline{\pi}^T A$ and $\underline{\pi}^T \underline{e} = 1$, then

$$\pi_1 = \frac{e^{-\mu x}-e^{-(\gamma+\mu)x}}{(1-e^{-(\gamma+\mu)x})}, \tag{4.11}$$

$$\pi_2 = \frac{1-e^{-\mu x}}{(1-e^{-(\gamma+\mu)x})}. \tag{4.12}$$

The condition of stabilty for the present queueing system is given by $\underline{\pi}^T \underline{\beta} < 1$, with

$$\beta_i = \sum_{k=1}^{\infty} k\,[a_{i1}(k) + a_{i2}(k)]$$

$$= -\lambda[F_{i1}^{\sim\,\prime}(0,x) + F_{i2}^{\sim\,\prime}(0,x)], \quad i=1,2.$$

Thus, we have, after evaluating $F_{ij}^{\sim\,\prime}(0,x)$, $i,j=1,2$, using equations (4.7)-(4.10),

$$\beta_1 = \lambda[(\frac{(\gamma+\mu)^2+\gamma^2}{2\gamma\mu(\gamma+\mu)})\,(e^{\gamma x}-e^{-\mu x}) - \frac{1}{2\gamma}\,(1-e^{-\mu x})], \tag{4.13}$$

and

$$\beta_2 = \lambda[(\frac{(\gamma+\mu)^2+\gamma^2}{2\gamma\mu(\gamma+\mu)})\,(e^{\gamma x}+e^{-(\gamma+\mu)x}-e^{-\mu x}-1) - \frac{1}{2\gamma}\,(e^{-(\gamma+\mu)x}-e^{-\mu x})]. \tag{4.14}$$

The condition of stability follows $(\pi_1\beta_1+\pi_2\beta_2 < 1)$,

$$\lambda < \frac{2\gamma\mu(\gamma+\mu)}{((\gamma+\mu)^2+\gamma^2)\,(e^{\gamma x}-1)} = \lambda^*. \tag{4.15}$$

Now we proceed to determine the mean of the number of jobs in the system, in steady state.

Let the irreducible stochastic matrix H (defined in section 3) be given by

$$H = \begin{bmatrix} \delta & 1-\delta \\ 1-\theta & \theta \end{bmatrix}.$$

It is determined as the smallest solution to

$$H = \int_0^\infty dF(t,x)e^{\lambda(H-I)t} \tag{4.16}$$

with $dF(t,x) = [dF_{ij}(t,x)]$, $i,j=1,2$.

It is easy to show that

$$e^{\lambda(H-I)t} = \frac{1}{\alpha}\begin{bmatrix} 1-\theta+(1-\delta)e^{-\alpha t} & 1-\delta+(\delta-1)e^{-\alpha t} \\ 1-\theta+(\theta-1)e^{-\alpha t} & 1-\delta+(1-\theta)e^{-\alpha t} \end{bmatrix}. \tag{4.17}$$

where $\alpha = \lambda(2-\delta-\theta)$. Hence substituting (4.17) in (4.16), we get:

$$H = \frac{1}{\alpha}\begin{bmatrix} (1-\theta)+(1-\delta)F^{*}_{11}(\alpha,x)+(\theta-1)F^{*}_{12}(\alpha,x) & (1-\delta)+(\delta-1)F^{*}_{11}(\alpha,x)+(1-\theta)F^{*}_{12}(\alpha,x) \\ (1-\theta)+(1-\delta)F^{*}_{21}(\alpha,x)+(\theta-1)F^{*}_{22}(\alpha,x) & (1-\delta)+(\delta-1)F^{*}_{21}(\alpha,x)+(1-\theta)F^{*}_{22}(\alpha,x) \end{bmatrix}$$

The unknowns δ and θ can be determined by solving the following two nonlinear equations

$$\delta\alpha = (1-\theta)+(1-\delta)F^{*}_{11}(\alpha,x)+(\theta-1)F^{*}_{12}(\alpha,x), \tag{4.18}$$

$$\theta\alpha = (1-\delta)+(1-\theta)F^{*}_{22}(\alpha,x)+(\delta-1)F^{*}_{21}(\alpha,x) \tag{4.19}$$

with $F^{*}_{ij}(\alpha,x)$, $i,j=1,2$, from equations (4.7)-(4.10).

Equations (4.18) and (4.19) are solved using Broyden's method [4] which converges quickly for judiciously chosen initial solution vectors. Then the vector $\underline{y}_0^T$ is solved for by using (3.4) and (3.6). Using $\underline{y}_0^T$ we then compute $\underline{\phi}^T(w)$ from (3.1). By the method described in section 3, we are then able to compute the expected number of jobs in the system in the steady state. We plot the expected number of jobs as a function of λ in figure 2 for $x=0.01, \mu=1$ for three different values of the failure rate $\gamma=0.01, 0.05, 0.1$. As expected, increasing the failure rate γ implies a substantial increase in system congestion.

In the following table, we give the expected service time in the steady state as a function of λ/λ^* for $\gamma=0.1$, $\mu=1$ and $x=0.01$. In this case the threshold arrival rate λ^* is 180.24. The expected service time of an arbitrary job in steady state is denoted by E(S).

λ/λ^*	$E(S)*10^3$
0.99	5.55
0.90	5.60
0.80	5.68
0.70	5.77
0.60	5.88
0.50	5.98
0.40	6.05
0.30	6.14
0.20	6.30
0.10	6.71
0.00	19.9

It is seen that the expected service time *reduces* from 0.0199 to 0.0055 as λ incereases from 0 to 0.99* λ^*. This seemingly non-intuitive result appears because as λ increases, the probability that a job will be taken up for service when both processors are down decreases.

References

[1] Baccelli, F. and Trivedi, K.S., "A Single Server Queue in a Hard-Real-Time Environment," *Operations Research Letters*, Vol. 4, No. 4, 1985.

[2] Baccelli, F. and Trivedi, K. S., "Analysis of M/G/2-Standby Redundant System," *Proc. 9-th IFIP International Symposium on Computer Performance Modeling, Performance '83*, A. K. Agrawala and S. k. Tripathi (eds.), North Holland, Amsterdam, 1983.

[3] Barlow, R. E. and Proschan, F., "Statistical Theory of Reliability and Life Testing: Probability Models," Holt, Rinehart and Winston, New York, 1975.

[4] Broyden, C.G., "A Class of Methods for Solving Nonlinear Simultaneous Equations," *Math. Comp.*, Vol. 19, pp. 577-593, 1965.

[5] Cooper, R. B., *"Introduction to Queueing Theory,"* 2-nd edittion, North Holland, 1981.

[6] Gaver D. P., "A Waiting Line with Interrupted Service, Including Priorities," *J. Royal Statistical Society*, Series B-24, 1962, pp. 73-90.

[7] Kleinrock, L., *"Queueing Systems,"* Vol. I & II, John Wiley & sons, 1976.

[8] Kulkarni, V. G., Nicola, V. F. and Trivedi, K. S., "On Modeling the Performance and Reliability of Multi-Mode Computer Systems," in M. Becker (ed.), *Proc. Int. Workshop on Modeling and Performance Evaluation of* Parallel Systems, North-Holland, 1984.

[9] Kulkarni, V. G., Nicola, V. F., Trivedi, K. S. and Smith, R. M., "A Unified Model for the Analysis of Job Completion Time and Performability Measures in Fault-Tolerant Systems," in review, *JACM*.

[10] Kulkarni, V. G., Nicola, V. F. and Trivedi, K. S., "The Completion Time of a Job on Multi-Mode Systems," in review, *J. of Applied Probability.*

[11] Lucantoni, D. M. and Neuts, M. F., "Numerical Methods for a Class of Markov Chains Arising in Queueing Theory," Tech. Report, No. 78/10, Dept. of Statistics and Computer Science, University of Delaware, 1978.

[12] Lucantoni, D. M. and Ramaswami, V., "Efficient Algorithms for Solving the Non-Linear Matrix Equations Arising in Phase Type Queues," *Stochastic Models*, Vol. 1, No. 1, 1985, pp. 29-52.

[13] Meyer, J., "Closed-Form Solution of Performability," *IEEE Transactions on Computers*, Vol. C-31, No. 7, July 1982, pp. 648-657.

[14] Mitrani, I. L. and Avi-Itzhak, B., "A Many-Server Queue with Service Interruptions," *Operations Research*, Vol. 16, 1968, pp. 628-638.

[15] Neuts, M. F., "Matrix Analytic Methods in Queueing Theory," *European Journal of Operations Research*, Vol. 15, pp. 2-12, 1984.

[16] Neuts, M.F. and Lucantoni, D.M., "A Markovian Queue with N Servers Subject to Breakdowns and Repairs," *Management Science*, Vol. 25, No. 9, pp. 849-861, 1979.

[17] Nicola, V. F., "A Single Server Queue with Mixed Types of Interruptions," in review, *Acta Informatica*.

[18] Purdue, P., "The M/M/1 Queue in a Markovian Environment," *Operations Research*, Vol. 22, 19 , pp. 562-569.

[19] Ramaswami, V., "The N/G/1 Queue and its Detailed Analysis," *Adv. Appl. Prob.*, Vol. 12, 1980, pp. 222-261.

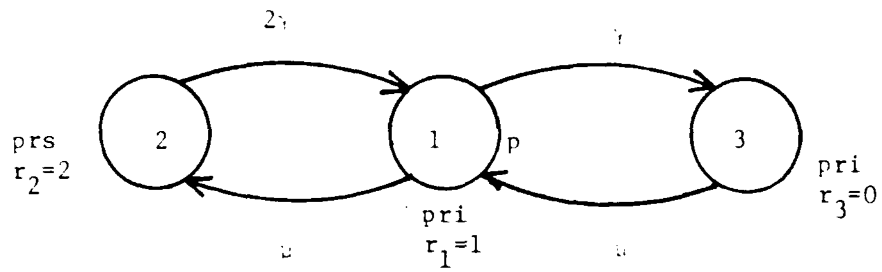

Figure 1

Rate Diagram For The Two Processor System

Expected Queue Length vs. Arrival Rate

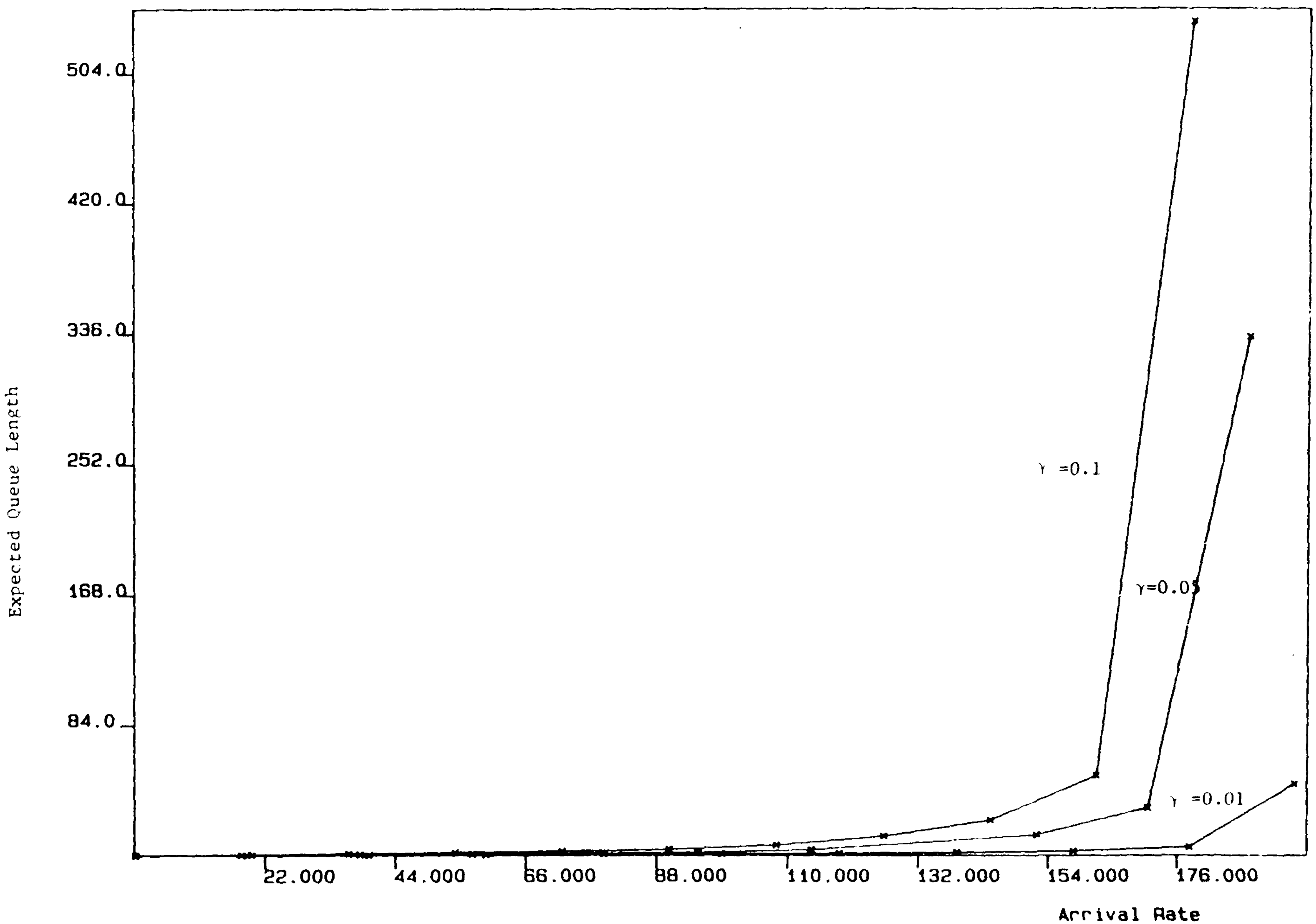

Figure 2

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